The A_∞ structure from the Berkovits formulation of open superstring field theory

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March 5, 2015
String Field Theory 2015
at Nara Women's University

Based on an upcoming paper
with Erler and Takezaki

1. Introduction

Can we consistently quantize string field theory?

- clue to fundamental degrees of freedom of string theory
- open strings versus closed strings

Open bosonic string field theory and closed bosonic string field theory have been quantized based on the Batalin-Vilkovisky formalism.

However, the quantization of the bosonic string is formal because of the presence of tachyons.

How about the quantization of superstring field theory?

Among various formulations of superstring field theory, the Berkovits formulation for the Nevue-Schwarz (NS) sector of open superstring field theory has been quite successful.

The Berkovits formulation is based on the large Hilbert space of the superconformal ghost sector.

The quantization based on the Batalin-Vilkovisky formalism, however, has turned out to be formidably complicated.

Why is it so complicated?

In bosonic string field theory, the equation of motion and the gauge transformation can both be written in terms of the same set of string products.

The string products satisfy the set of relations called A_∞ for the open string and L_∞ for the closed string.

These structures play a crucial role in the Batalin-Vilkovisky quantization, and they are closely related to the decomposition of the moduli space of Riemann surfaces.

The source of the difficulty for the Batalin-Vilkovisky quantization in the Berkovits formulation can be seen in the free theory.

The equation of motion:

$$Q\eta\Phi = 0\,.$$

Φ: the open superstring field in the large Hilbert space,
Q: the BRST operator,
η: the zero mode of the superconformal ghost $\eta(z)$.

The gauge transformations:

$$\delta\Phi = Q\Lambda + \eta\Omega\,.$$

Λ, Ω: the gauge parameters.

The difference of the structure between $Q\eta\Phi = 0$ and $\delta\Phi = Q\Lambda + \eta\Omega$ can be thought of as the source of the difficulty.

Working in the large Hilbert space obscures the relation to the supermoduli space of super-Riemann surfaces, and it might be one possible reason underlying the difficulty.

Approaches to incorporating the Ramond sector are also complicated (see the talk by Kunitomo san), and it might also be related to our insufficient understanding of the connection between the large Hilbert space and the supermoduli space of super-Riemann surfaces.

What was the difficulty in formulating open superstring field theory based on the small Hilbert space? Consider the free theory.

The equation of motion: $Q\Psi = 0\,.$

Ψ: the open superstring field in the small Hilbert space

The gauge transformation: $\delta\Psi = Q\Lambda\,.$

Λ: the gauge parameter.

They are both written in terms of Q, and this seems to be promising for constructing string products satisfying the A_∞ relations.

It had long been thought, however, that a regular formulation based on the small Hilbert space would be difficult because of singularities coming from local picture-changing operators.

Recently, it was demonstrated that a regular formulation based on the small Hilbert space can be obtained from the Berkovits formulation by partial gauge fixing.

Iimori, Noumi, Okawa and Torii, arXiv:1312.1677

New ingredient: an operator ξ satisfying $\{\eta, \xi\} = 1$. Such an operator can be realized by a line integral of the superconformal ghost $\xi(z)$.

- The partial gauge fixing guarantees that the resulting theory is gauge invariant.
- The BRST transformation of ξ yields a line integral of the picture-changing operator, and singularities associated with local picture-changing operators are avoided in this approach.
- However, it turned out that the resulting theory does not exhibit the A_∞ structure.

Once we recognize that ξ can be used in constructing a guage-invariant action, we do not necessarily start from the Berkovits formulation.

Erler, Konopka and Sachs constructed an action with the A_∞ structure for the NS sector of open superstring field theory based on the small Hilbert space using ξ as a new ingredient.

Erler, Konopka and Sachs, arXiv:1312.2948

- Because of the A_∞ structure, the Batalin-Vilkovisky quantization is straightforward.
- The construction was further generalized to the NS sector of heterotic string field theory and the NS-NS sector of type II superstring field theory. (See the talk by Matsunaga kun.)

Erler, Konopka and Sachs, arXiv:1403.0940

We now have two successful formulations for the NS sector of open superstring field theory.

- The theory by Berkovits is beautifully formulated based on the large Hilbert space.
- The theory by Erler, Konopka and Sachs is based on the small Hilbert space and exhibits the A_∞ structure.

In this talk, we show that the two theories are related by field redefinition and partial gauge fixing.

The talk is based on an upcoming paper with Erler and Takezaki.

The plan of the talk

- 1. Introduction
- 2. Open bosonic string field theory
- 3. The Witten and Berkovits formulations
- 4. New formulations in the small Hilbert space
- 5. The A_∞ structure from the Berkovits formulation

2. Open bosonic string field theory

Open bosonic string field theory

Witten, Nucl. Phys. B268 (1986) 253

$$S = -\frac{1}{2}\langle \Psi, Q\Psi \rangle - \frac{g}{3}\langle \Psi, \Psi * \Psi \rangle .$$

- Ψ: the open bosonic string field
- Q: the BRST operator
- g: the open string coupling constant
- $\langle A, B \rangle$: the BPZ inner product
- $A * B$: star product
  noncommutative $A * B \neq B * A$
  but associative $(A * B) * C = A * (B * C)$

The action is invariant under the gauge transformation given by

$$\delta_\Lambda \Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi \,,$$

where Λ is the gauge parameter

The invariance can be shown only from

$$\begin{aligned}
\langle\, B, A \,\rangle &= (-1)^{AB} \,\langle\, A, B \,\rangle \,, \\
Q^2 &= 0 \,, \\
\langle\, QA, B \,\rangle &= \, -\,(-1)^{A} \langle\, A, QB \,\rangle \,, \\
\langle\, A, B * C \,\rangle &= \langle\, A * B, C \,\rangle \,, \\
(A * B) * C &= A * (B * C) \,, \\
Q\,(A * B) &= QA * B + (-1)^{A} A * QB \,.
\end{aligned}$$

Actually, we can construct a gauge-invariant action based on a string product without associativity. Consider an action in the following form:

$$S = -\frac{1}{2}\langle \Psi, Q\Psi \rangle - \frac{g}{3}\langle \Psi, V_2(\Psi, \Psi) \rangle - \frac{g^2}{4}\langle \Psi, V_3(\Psi, \Psi, \Psi) \rangle + O(g^3)\,.$$

The BRST operator Q can be thought of as a one-string product with the cyclic property

$$\langle A_1, QA_2 \rangle = -(-1)^{A_1}\langle QA_1, A_2 \rangle\,.$$

$V_2(A_1, A_2)$: two-string product with the cyclic property

$$\langle A_1, V_2(A_2, A_3) \rangle = \langle V_2(A_1, A_2), A_3 \rangle\,.$$

$V_3(A_1, A_2, A_3)$: three-string product with the cyclic property

$$\langle A_1, V_3(A_2, A_3, A_4) \rangle = -(-1)^{A_1}\langle V_3(A_1, A_2, A_3), A_4 \rangle\,.$$

The action is invariant up to $O(g^3)$,

$$\delta_\Lambda S = O(g^3)\,,$$

under the gauge transformation in the form

$$\begin{aligned}\delta_\Lambda \Psi = Q\Lambda + g\,\big(\,V_2(\Psi,\Lambda) - V_2(\Lambda,\Psi)\,\big) \\ + g^2\,\big(\,V_3(\Psi,\Psi,\Lambda) - V_3(\Psi,\Lambda,\Psi) + V_3(\Lambda,\Psi,\Psi)\,\big) + O(g^3)\end{aligned}$$

if Q, V_2, and V_3 satisfy

$$\begin{aligned}& Q^2 A_1 = 0\,, \\ & QV_2(A_1,A_2) - V_2(QA_1,A_2) - (-1)^{A_1} V_2(A_1,QA_2) = 0\,, \\ & QV_3(A_1,A_2,A_3) - V_2(V_2(A_1,A_2),A_3) + V_2(A_1,V_2(A_2,A_3)) \\ & + V_3(QA_1,A_2,A_3) + (-1)^{A_1} V_3(A_1,QA_2,A_3) \\ & + (-1)^{A_1+A_2} V_3(A_1,A_2,QA_3) = 0\,.\end{aligned}$$

- These relations of multi-string products are extended to higher orders, and a set of these relations is called the A_∞ structure.
- The A_∞ structure is closely related to the decomposition of the moduli space of Riemann surfaces.
- The quantization of string field theory based on the Batalin-Vilkovisky formalism is straightforward if the theory has the A_∞ structure.

3. The Witten and Berkovits formulations

In the Ramond-Neveu-Schwarz formalism of the superstring, there are infinitely many ways to describe each physical state, and they are labeled by a quantum number called *picture*.

In tree-level scattering amplitudes of the open superstring, the sum of the picture numbers of external states has to be -2.

e.g. four-point amplitudes of bosons
We can choose two states to be in the -1 picture and two states to be in the 0 picture.

On-shell scattering amplitudes do not depend on a choice of pictures, but how should we deal with the picture in string field theory?

Physical states in different pictures are mapped by the picture-changing operator $X(z)$:

$$\Psi^{(0)}(w) = \lim_{z \to w} X(z)\, \Psi^{(-1)}(w)\,,$$

where

$$[\, Q, X(z)\,] = 0\,.$$

e.g. four-point amplitudes
We can choose all the four states to be in the -1 picture and insert two picture-changing operators.

The Witten formulation

Nucl. Phys. B276 (1986) 291

The open superstring field is in the -1 picture.

Choose the two-string product to be

$$V_2(A_1, A_2) = X_{\rm mid}\,(A_1 * A_2)\,.$$

$X_{\rm mid}$: the picture-changing operator inserted at the open-string midpoint.

The operator product expansion of two picture-changing operators is singular.

→ divergences in the gauge variation of the action and in four-point amplitudes

The Berkovits formulation   hep-th/9503099

open superstring field:
a state in the matter + bc ghost + superconformal ghost CFT

the superconformal ghost sector
$\beta(z)\,\gamma(z) \quad \rightarrow \quad \xi(z),\, \eta(z), \phi(z)$

The Hilbert space for $\xi(z),\, \eta(z), \phi(z)$ is larger and is called the large Hilbert space.

The Hilbert space we usually use for $\beta\gamma$ ghosts is called the small Hilbert space.

$$\Psi \in \text{the small Hilbert space} \iff \eta\Psi = 0$$

η: the zero mode of $\eta(z)$

Algebraic relations in the large Hilbert space

$$\langle B, A \rangle = (-1)^{AB} \langle A, B \rangle \,,$$

$$Q^2 = 0 \,, \quad \eta^2 = 0 \,, \quad \{ Q, \eta \} = 0 \,,$$

$$\langle QA, B \rangle = -(-1)^{A} \langle A, QB \rangle \,,$$

$$\langle \eta A, B \rangle = -(-1)^{A} \langle A, \eta B \rangle \,,$$

$$\langle A, B * C \rangle = \langle A * B, C \rangle \,,$$

$$(A * B) * C = A * (B * C) \,,$$

$$Q\,(A * B) = QA * B + (-1)^{A} A * QB \,,$$

$$\eta\,(A * B) = \eta A * B + (-1)^{A} A * \eta B \,.$$

How large is the large Hilbert space?

$$\eta^2 = 0 \quad \text{and} \quad {}^{\exists}\xi \quad \text{satisfying} \quad \{\, \eta, \xi \,\} = 1 \,.$$

A state Φ in the large Hilbert space can be decomposed as follows:

$$\Phi = \eta\xi\Phi + \xi\eta\Phi = \Psi_1 + \xi\Psi_2 \,.$$

$$\Psi_1, \Psi_2 \in \text{the small Hilbert space}$$

We could say that the large Hilbert space is twice as large as the small Hilbert space.

We can realize ξ by a line integral of $\xi(z)$, and we assume that ξ obeys $\xi^2 = 0$ and $\langle\, A, \xi B \,\rangle = (-1)^A \langle\, \xi A, B \,\rangle$. We can choose, for example, ξ to be the zero mode ξ_0 of $\xi(z)$.

The BPZ inner products

For a pair of string fields A and B in the small Hilbert space, we define $\langle\!\langle\, A, B \,\rangle\!\rangle$ by

$$\langle\!\langle\, A, B \,\rangle\!\rangle = \langle\, \xi_0 A, B \,\rangle \,.$$

We can use ξ to relate the two BPZ inner products as

$$\langle\!\langle\, A, B \,\rangle\!\rangle = \langle\, \xi A, B \,\rangle \,.$$

The action of the free theory in the Berkovits formulation:

$$S = -\frac{1}{2}\langle \Phi, Q\eta\Phi \rangle .$$

Φ: the open superstring field in the large Hilbert space

The equation of motion: $Q\eta\Phi = 0$.
The gauge transformations: $\delta\Phi = Q\Lambda + \eta\Omega$.

Λ, Ω: gauge parameters in the large Hilbert space

By the gauge transformation $\delta\Phi = \eta\Omega$,

$$\Phi = \eta\xi\Phi + \xi\eta\Phi \quad \Longrightarrow \quad \Phi = \xi\Psi$$

$\Psi \in$ the small Hilbert space

The equation of motion reduces to

$$Q\eta\Phi = Q\eta\xi\Psi = Q\{\eta, \xi\}\Psi = Q\Psi = 0 .$$

The action of the interacting theory:

$$S = -\frac{1}{2}\langle\, \Phi, Q\eta\Phi \,\rangle + \frac{g}{6}\langle\, \eta\Phi, [\, \Phi, Q\Phi \,] \,\rangle$$
$$- \frac{g^2}{24}\langle\, \eta\Phi, [\, \Phi, [\, \Phi, Q\Phi \,]\,] \,\rangle + O(g^3)\,.$$

The nonlinearly extended gauge transformations:

$$\delta\Phi = Q\Lambda - \frac{g}{2}[\, \Phi, Q\Lambda \,] + \frac{g^2}{12}[\, \Phi, [\, \Phi, Q\Lambda \,]\,]$$
$$+ \eta\Omega + \frac{g}{2}[\, \Phi, \eta\Omega \,] + \frac{g^2}{12}[\, \Phi, [\, \Phi, \eta\Omega \,]\,] + O(g^3)\,.$$

(Here and in what follows, we often omit the star symbol for the star product.)

The action to all orders in g is written in a Wess-Zumino-Witten-like (WZW-like) form.

- The correct four-point amplitudes of bosons are reproduced.
- The Ramond sector: considerably complicated and not completely covariant (covariant, however, for a class of interesting backgrounds such as D3-branes) See also the talk by Kunitomo san.
- The Batalin-Vilkovisky formalism for quantization: formidably complicated

Working in the large Hilbert space obscures the relation to the supermoduli space of super-Riemann surfaces.

4. New formulations in the small Hilbert space

Partial gauge fixing

Iimori, Noumi, Okawa and Torii, arXiv:1312.1677

In the Berkovits formulation, we can impose the following condition for partial gauge fixing in the interacting theory:

$$\xi \Phi = 0 \,.$$

The open superstring field Φ satisfying this condition can be written as

$$\Phi = \xi \Psi \quad \text{with} \quad \Psi \in \text{the small Hilbert space.}$$

Replace Φ in the action $S[\,\Phi\,]$ of the Berkovits formulation by $\xi\Psi$, and regard $S[\,\xi\Psi\,]$ as an action for Ψ. The resulting theory for Ψ is guaranteed to be invariant under the residual gauge transformation.

→ This provides a consistent formulation based on the small Hilbert space!

When $Q\Psi = 0$, the cubic interaction

$$\frac{1}{6}\langle\,\Psi, \xi\Psi * Q\xi\Psi\,\rangle - \frac{1}{6}\langle\,\Psi, Q\xi\Psi * \xi\Psi\,\rangle$$

reduces to

$$-\frac{1}{3}\langle\!\langle\, X\Psi, \Psi * \Psi\,\rangle\!\rangle\,,$$

where $X = \{\,Q, \xi\,\}$, which is a line integral of the picture-changing operator $X(z)$

We have learned that we can use a line integral ξ in the construction of a gauge-invariant action to obtain a regular formulation without singularities coming from local picture-changing operators.

The resulting theory based on the small Hilbert space, however, does not exhibit the A_∞ structure.

Open superstring field theory with the A_∞ structure

Erler, Konopka and Sachs, arXiv:1312.2948

The two-string product

$$V_2(A_1, A_2) = \frac{1}{3}\Big[\, X(A_1 A_2) + (XA_1)A_2 + A_1(XA_2) \Big]$$

satisfies

$$\langle\, A_1, V_2(A_2, A_3)\,\rangle = \langle\, V_2(A_1, A_2), A_3)\,\rangle$$

and

$$QV_2(A_1, A_2) - V_2(QA_1, A_2) - (-1)^{A_1} V_2(A_1, QA_2) = 0\,,$$

but V_2 is not associative. We need a three-string product V_3 satisfying

$$\begin{aligned} &QV_3(A_1, A_2, A_3) - V_2(V_2(A_1, A_2), A_3) + V_2(A_1, V_2(A_2, A_3)) \\ &+ V_3(QA_1, A_2, A_3) + (-1)^{A_1} V_3(A_1, QA_2, A_3) \\ &+ (-1)^{A_1+A_2} V_3(A_1, A_2, QA_3) = 0\,. \end{aligned}$$

Erler, Konopka and Sachs constructed $V_3(A_1, A_2, A_3)$ satisfying this relation from the star product, ξ, and X.

Furthermore, they extended the construction to higher orders and succeeded in constructing an action of open superstring field theory based on the small Hilbert space with the A_∞ structure!

How did they construct multi-string products satisfying the A_∞ relations?

The two-string product can be written as

$$V_2(A_1, A_2) = Q V_2^\xi(A_1, A_2) + V_2^\xi(QA_1, A_2) + (-1)^{A_1} V_2^\xi(A_1, QA_2)$$

with

$$V_2^\xi(A_1, A_2) = \frac{1}{3}\Big[\, \xi(A_1 A_2) + (\xi A_1) A_2 + (-1)^{A_1} A_1 (\xi A_2) \Big] .$$

In general, if the two-string product $V_2(A_1, A_2)$ is written as

$$V_2(A_1, A_2) = QV_2^\xi(A_1, A_2) + V_2^\xi(QA_1, A_2) + (-1)^{A_1} V_2^\xi(A_1, QA_2)\,,$$

and $V_2^\xi(A_1, A_2)$ satisfies

$$\langle\, A_1, V_2^\xi(A_2, A_3)\,\rangle = (-1)^{A_1}\,\langle\, V_2^\xi(A_1, A_2), A_3)\,\rangle\,,$$

the conditions

$$\langle\, A_1, V_2(A_2, A_3)\,\rangle = \langle\, V_2(A_1, A_2), A_3)\,\rangle$$

and

$$QV_2(A_1, A_2) - V_2(QA_1, A_2) - (-1)^{A_1} V_2(A_1, QA_2) = 0$$

are satisfied.

However, the cubic interaction

$$-\frac{1}{3}\langle\!\langle\, \Psi, V_2(\Psi, \Psi)\,\rangle\!\rangle$$

can be generated from the free theory by field redefinition if $\eta V_2^\xi(\Psi, \Psi) = 0\,.$

To obtain an interacting theory, we need

$$\eta V_2^\xi(A_1, A_2) + V_2^\xi(\eta A_1, A_2) + (-1)^{A_1} V_2^\xi(A_1, \eta A_2) \neq 0\,,$$

while the condition

$$\eta V_2(A_1, A_2) - V_2(\eta A_1, A_2) - (-1)^{A_1} V_2(A_1, \eta A_2) = 0$$

is satisfied.

This can be achieved if V_2^ξ satisfies

$$\eta V_2^\xi(A_1, A_2) + V_2^\xi(\eta A_1, A_2) + (-1)^{A_1} V_2^\xi(A_1, \eta A_2) = A_1 A_2 \,,$$

and $V_2^\xi(A_1, A_2)$ satisfying this relation can be constructed using ξ because the cohomology of η is trivial.

As we said before, the two-string product V_2 is not associative and we need V_3 satisfying

$$\begin{aligned} &QV_3(A_1, A_2, A_3) - V_2(V_2(A_1, A_2), A_3) + V_2(A_1, V_2(A_2, A_3)) \\ &+ V_3(QA_1, A_2, A_3) + (-1)^{A_1} V_3(A_1, QA_2, A_3) \\ &+ (-1)^{A_1+A_2} V_3(A_1, A_2, QA_3) = 0 \,. \end{aligned}$$

This relation is satisfied if V_3 is written in terms of V_2^ξ and $V_3^{\xi\xi}$ as

$$
\begin{aligned}
& V_3(A_1, A_2, A_3) \\
& = \frac{1}{2} \Big[Q V_3^{\xi\xi}(A_1, A_2, A_3) - V_3^{\xi\xi}(Q A_1, A_2, A_3) \\
& \qquad - (-1)^{A_1} V_3^{\xi\xi}(A_1, Q A_2, A_3) - (-1)^{A_1 + A_2} V_3^{\xi\xi}(A_1, A_2, Q A_3) \\
& \qquad + Q V_2^\xi(V_2^\xi(A_1, A_2), A_3) - (-1)^{A_1} Q V_2^\xi(A_1, V_2^\xi(A_2, A_3)) \\
& \qquad + 2\, V_2^\xi(Q V_2^\xi(A_1, A_2), A_3) - 2\, V_2^\xi(A_1, Q V_2^\xi(A_2, A_3)) \\
& \qquad + V_2^\xi(V_2^\xi(Q A_1, A_2), A_3) + (-1)^{A_1} V_2^\xi(V_2^\xi(A_1, Q A_2), A_3) \\
& \qquad - (-1)^{A_1 + A_2} V_2^\xi(V_2^\xi(A_1, A_2), Q A_3) - (-1)^{A_1} V_2^\xi(Q A_1, V_2^\xi(A_2, A_3)) \\
& \qquad - V_2^\xi(A_1, V_2^\xi(Q A_2, A_3)) - (-1)^{A_2} V_2^\xi(A_1, V_2^\xi(A_2, Q A_3)) \Big] \, .
\end{aligned}
$$

To ensure that the condition

$$\eta V_3(A_1, A_2, A_3) + V_3(\eta A_1, A_2, A_3) + (-1)^{A_1} V_3(A_1, \eta A_2, A_3)$$
$$+ (-1)^{A_1+A_2} V_3(A_1, A_2, \eta A_3) = 0$$

is satisfied, V_2^{ξ} and $V_3^{\xi\xi}$ have to be related as

$$\eta V_3^{\xi\xi}(A_1, A_2, A_3) - V_3^{\xi\xi}(\eta A_1, A_2, A_3)$$
$$- (-1)^{A_1} V_3^{\xi\xi}(A_1, \eta A_2, A_3) - (-1)^{A_1+A_2} V_3^{\xi\xi}(A_1, A_2, \eta A_3)$$
$$= V_2^{\xi}(A_1, A_2)\, A_3 - (-1)^{A_1} A_1\, V_2^{\xi}(A_2, A_3)$$
$$+ V_2^{\xi}(A_1 A_2, A_3) - V_2^{\xi}(A_1, A_2 A_3)\,.$$

We can construct $V_3^{\xi\xi}(A_1, A_2, A_3)$ satisfying this relation because the cohomology of η is trivial.

For V_2^{ξ} given by

$$V_2^{\xi}(A_1, A_2) = \frac{1}{3}\Big[\xi(A_1 A_2) + (\xi A_1)A_2 + (-1)^{A_1} A_1(\xi A_2)\Big],$$

one realization of $V_3^{\xi\xi}$ is

$$\begin{aligned} & V_3^{\xi\xi}(A_1, A_2, A_3) \\ & = \frac{1}{6}\Big[\xi((\xi(A_1 A_2))\, A_3) - (-1)^{A_1}\xi(A_1\,(\xi(A_2 A_3))) \\ & \qquad - (\xi((\xi A_1)A_2)))\, A_3 - (-1)^{A_1}(\xi A_1)\,(\xi(A_2 A_3)) \\ & \qquad - (-1)^{A_1}(\xi(A_1(\xi A_2)))\, A_3 + A_1\,(\xi((\xi A_2)A_3)) \\ & \qquad - (-1)^{A_1+A_2}(\xi(A_1 A_2))\,(\xi A_3) + (-1)^{A_2} A_1\,(\xi(A_2(\xi A_3)))\Big]. \end{aligned}$$

This construction can be extended to all orders.

What is the relation to the Berkovits formulation based on the large Hilbert space?

5. The A_∞ structure from the Berkovits formulation

The WZW-like action of the Berkovits formulation is

$$
\begin{aligned}
S = & -\frac{1}{2}\langle\, e^{-\Phi}(\eta e^{\Phi}), e^{-\Phi}(Q e^{\Phi})\,\rangle \\
& -\int_0^1 dt\,\langle\, e^{-\Phi(t)}\partial_t e^{\Phi(t)}, \{\, e^{-\Phi(t)}(Q e^{\Phi(t)}), e^{-\Phi(t)}(\eta e^{\Phi(t)})\,\}\,\rangle\,,
\end{aligned}
$$

where $\Phi(1) = \Phi$ and $\Phi(0) = 0$. This can also be written as

$$
S = -\int_0^1 dt\,\langle\, A_t(t), Q A_\eta(t)\,\rangle
$$

with

$$
A_\eta(t) = (\eta\, e^{\Phi(t)})\, e^{-\Phi(t)}\,, \quad A_t(t) = (\partial_t e^{\Phi(t)})\, e^{-\Phi(t)}\,.
$$

The t-dependence is topological:

$$\delta \langle\, A_t(t), QA_\eta(t) \,\rangle = \partial_t \langle\, A_t(t), QA_\eta(t) \,\rangle \,.$$

The topological t-dependence and the gauge invariance follow from

$$\eta A_\eta(t) = A_\eta(t) A_\eta(t) \,,$$
$$\partial_t A_\eta(t) = \eta A_t(t) - A_\eta(t) A_t(t) + A_t(t) A_\eta(t)$$

together with the fact that the cohomology of η is trivial.

To describe the A_∞ structure, it is convenient to introduce *degree* for a string field A denoted by $\deg(A)$. It is defined by

$$\deg(A) = \epsilon(A) + 1 \quad \mathrm{mod}\ 2\,,$$

where $\epsilon(A)$ is the Grassmann parity of A.

We define $\omega(A_1, A_2)$, $\omega_L(A_1, A_2)$, $M_2(A_1, A_2)$, $M_3(A_1, A_2, A_3)$ by

$$\begin{aligned}
\omega(A_1, A_2) &= (-1)^{\deg(A_1)} \langle\!\langle A, B \rangle\!\rangle\,, \\
\omega_L(A_1, A_2) &= (-1)^{\deg(A_1)} \langle A, B \rangle\,, \\
M_2(A_1, A_2) &= (-1)^{\deg(A_1)}\, V_2(A_1, A_2)\,, \\
M_3(A_1, A_2, A_3) &= (-1)^{\deg(A_2)}\, V_3(A_1, A_2, A_3)\,.
\end{aligned}$$

It is further convenient to introduce linear operators called *coderivatons* acting on the vector space $T\mathcal{H}$ defined by

$$T\mathcal{H} = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \ldots \,,$$

where $\mathcal{H}$ is the Hilbert space.

String products are promoted to coderivations as follows. For the BRST operator as a one-string product, we have

$$\begin{aligned}
\mathbf{Q}\,1 &= 0\,,\\
\mathbf{Q}\,A_1 &= QA_1\,,\\
\mathbf{Q}\,(\,A_1 \otimes A_2\,) &= QA_1 \otimes A_2 + (-1)^{\deg(A_1)} A_1 \otimes QA_2\,,\\
\mathbf{Q}\,(\,A_1 \otimes A_2 \otimes A_3\,) &= QA_1 \otimes A_2 \otimes A_3 + (-1)^{\deg(A_1)} A_1 \otimes QA_2 \otimes A_3\\
&\quad + (-1)^{\deg(A_1)+\deg(A_2)} A_1 \otimes A_2 \otimes QA_3\,,\\
&\;\;\vdots
\end{aligned}$$

For the two-string product M_2, we have

$$\begin{aligned}
\mathbf{M}_2\, 1 &= 0\,, \\
\mathbf{M}_2\, A_1 &= 0\,, \\
\mathbf{M}_2\,(\, A_1 \otimes A_2\,) &= M_2(A_1, A_2)\,, \\
\mathbf{M}_2\,(\, A_1 \otimes A_2 \otimes A_3\,) &= M_2(A_1, A_2) \otimes A_3 \\
&\quad + (-1)^{\deg(A_1)} A_1 \otimes M_2(A_2, A_3)\,, \\
\mathbf{M}_2\,(\, A_1 \otimes A_2 \otimes A_3 \otimes A_4\,) &= M_2(A_1, A_2) \otimes A_3 \otimes A_4 \\
&\quad + (-1)^{\deg(A_1)} A_1 \otimes M_2(A_2, A_3) \otimes A_4\,, \\
&\quad + (-1)^{\deg(A_1)+\deg(A_2)} A_1 \otimes A_2 \otimes M_2(A_3, A_4)\,, \\
&\;\;\vdots
\end{aligned}$$

For the three-string product M_3, we have

$$
\begin{aligned}
\mathbf{M}_3\, 1 &= 0\,, \\
\mathbf{M}_3\, A_1 &= 0\,, \\
\mathbf{M}_3\, (\, A_1 \otimes A_2\,) &= 0\,, \\
\mathbf{M}_3\, (\, A_1 \otimes A_2 \otimes A_3\,) &= M_3(A_1, A_2, A_3)\,, \\
\mathbf{M}_3\, (\, A_1 \otimes A_2 \otimes A_3 \otimes A_4\,) &= M_3(A_1, A_2, A_3) \otimes A_4 \\
&\quad + (-1)^{\deg(A_1)} A_1 \otimes M_3(A_2, A_3, A_4)\,, \\
&\vdots
\end{aligned}
$$

We introduce $\mathbf{M}(s)$ defined by

$$\mathbf{M}(s) = \sum_{n=0}^{\infty} s^n \, \mathbf{M}_{n+1} \quad \text{with} \quad \mathbf{M}_1 = \mathbf{Q} \, .$$

Then the A_∞ relations can be compactly written as

$$[\, \mathbf{M}(s), \mathbf{M}(s) \,] = 0 \, .$$

The condition that $M_n(\Psi, \Psi, \ldots, \Psi)$ is in the small Hilbert space can be stated as

$$[\, \boldsymbol{\eta}, \mathbf{M}(s) \,] = 0 \, .$$

We construct $\mathbf{M}(s)$ from the star product. We define $m_2(A_1, A_2)$ by

$$m_2(A_1, A_2) = (-1)^{\deg(A_1)} A_1 A_2 \,,$$

and the corresponding coderivation $\mathbf{m}_2$ satisfy

$$[\, \mathbf{m}_2, \mathbf{m}_2 \,] = 0 \,, \qquad [\, \mathbf{Q}, \mathbf{m}_2 \,] = 0 \,, \qquad [\, \boldsymbol{\eta}, \mathbf{m}_2 \,] = 0 \,.$$

In arXiv:1312.2948 by Erler, Konopka and Sachs, $\mathbf{M}(s)$ was characterized by the following differential equation:

$$\frac{d}{ds} \mathbf{M}(s) = [\, \mathbf{M}(s), \boldsymbol{\mu}(s) \,] \quad \text{with} \quad \mathbf{M}(0) = \mathbf{Q} \,.$$

Its solution is

$$\mathbf{M}(s) = \mathbf{G}^{-1}(s) \, \mathbf{Q} \, \mathbf{G}(s) \,,$$

where $\mathbf{G}(s)$ is the path-ordered exponential given by

$$\mathbf{G}(s) = \mathcal{P} \exp \left[\int_0^s ds' \, \boldsymbol{\mu}(s') \right] .$$

If $[\,\boldsymbol{\eta}, \boldsymbol{\mu}(s)\,] = 0$, the resulting theory is related to the free theory by field redefinition. To obtain a nontrivial interacting theory, we need $[\,\boldsymbol{\eta}, \boldsymbol{\mu}(s)\,] \neq 0$, while $[\,\boldsymbol{\eta}, \mathbf{M}(s)\,] = 0$ is satisfied.

In arXiv:1312.2948 by Erler, Konopka and Sachs, $\boldsymbol{\mu}(s)$ was characterized by

$$[\,\boldsymbol{\eta}, \boldsymbol{\mu}(s)\,] = \mathbf{m}(s)\,,$$

and

$$\frac{d}{ds}\,\mathbf{m}(s) = [\,\mathbf{m}(s), \boldsymbol{\mu}(s)\,] \quad \text{with} \quad \mathbf{m}(0) = \mathbf{m}_2\,.$$

From these relations, we can show $[\,\boldsymbol{\eta}, \mathbf{G}(s)\,] = s\,\mathbf{m}_2\,\mathbf{G}(s)$, and the condition $[\,\boldsymbol{\eta}, \mathbf{M}(s)\,] = 0$ is shown to be satisfied.

To summarize, we have the following important relations:

$$\mathbf{M} = \mathbf{G}^{-1}\,\mathbf{Q}\,\mathbf{G}\,, \quad [\,\boldsymbol{\eta}, \mathbf{G}\,] = \mathbf{m}_2\,\mathbf{G}\,,$$

where $\mathbf{M} = \mathbf{M}(1)$, $\mathbf{G} = \mathbf{G}(1)$.

The action with the A_∞ structure can be written as

$$\begin{aligned} S = & -\frac{1}{2}\,\omega(\Psi, Q\Psi) - \frac{1}{3}\,\omega(\Psi, M_2(\Psi,\Psi)) - \frac{1}{4}\,\omega(\Psi, M_3(\Psi,\Psi,\Psi)) + \ldots \\ = & -\int_0^1 dt \left[\omega(\Psi, Q\,t\Psi) + \omega(\Psi, M_2(t\Psi, t\Psi)) + \omega(\Psi, M_3(t\Psi, t\Psi, t\Psi)) + \ldots\right] \\ = & -\int_0^1 dt\,\omega\Big(\,\Psi, \pi_1 \mathbf{M}\,\frac{1}{1-t\Psi}\,\Big)\,, \end{aligned}$$

where

$$\frac{1}{1-\Psi} = \sum_{n=0}^{\infty} \underbrace{\Psi \otimes \Psi \otimes \ldots \otimes \Psi}_{n} = 1 + \Psi + \Psi\otimes\Psi + \Psi\otimes\Psi\otimes\Psi \ldots$$

and π_1 is the projector to the one-string sector.

In terms of ω_L in the large Hilbert space, we have

$$S = \int_0^1 dt\, \omega_L\Big(\, \xi\Psi, \pi_1 \mathbf{M}\, \frac{1}{1-t\Psi}\,\Big)$$
$$= \int_0^1 dt\, \omega_L\Big(\, \pi_1\, \boldsymbol{\xi}_t\, \frac{1}{1-t\Psi}, \pi_1 \mathbf{M}\, \frac{1}{1-t\Psi}\,\Big)\,,$$

where $\boldsymbol{\xi}_t$ is the coderivation corresponding to $\xi\partial_t$.

When $\boldsymbol{\mu}(s)$ has an appropriate cyclic property, we find

$$S = \int_0^1 dt\, \omega_L\Big(\, \pi_1\, \boldsymbol{\xi}_t\, \frac{1}{1-t\Psi}, \pi_1\, \mathbf{G}^{-1}\, \mathbf{Q}\, \mathbf{G}\, \frac{1}{1-t\Psi}\,\Big)$$
$$= \int_0^1 dt\, \omega_L\Big(\, \pi_1\, \mathbf{G}\, \boldsymbol{\xi}_t\, \frac{1}{1-t\Psi}, Q\, \pi_1 \mathbf{G}\, \frac{1}{1-t\Psi}\,\Big)\,.$$

The action now takes the form

$$S = -\int_0^1 dt\,\langle\, A_t(t), QA_\eta(t)\,\rangle$$

with

$$A_\eta(t) = \pi_1\,\mathbf{G}\,\frac{1}{1-t\Psi}\,,\quad A_t(t) = \pi_1\,\mathbf{G}\,\boldsymbol{\xi}_t\,\frac{1}{1-t\Psi}\,,$$

and we can show that $A_\eta(t)$ and $A_t(t)$ satisfy

$$\begin{aligned}\eta A_\eta(t) &= A_\eta(t)A_\eta(t)\,,\\ \partial_t A_\eta(t) &= \eta A_t(t) - A_\eta(t)A_t(t) + A_t(t)A_\eta(t)\,.\end{aligned}$$

We can further show

$$A_\eta(t) = \pi_1 \, \mathbf{G} \, \frac{1}{1 - \Psi(t)} \,, \quad A_t(t) = \pi_1 \, \mathbf{G} \, \boldsymbol{\xi}_t \, \frac{1}{1 - \Psi(t)}$$

satisfy

$$\begin{aligned} \eta A_\eta(t) &= A_\eta(t) A_\eta(t) \,, \\ \partial_t A_\eta(t) &= \eta A_t(t) - A_\eta(t) A_t(t) + A_t(t) A_\eta(t) \,. \end{aligned}$$

- The dependence on t is topological.
- We can show that the theory with the A_∞ structure is related by field redefinition to the theory obtained from the Berkovits formulation by partial gauge fixing.

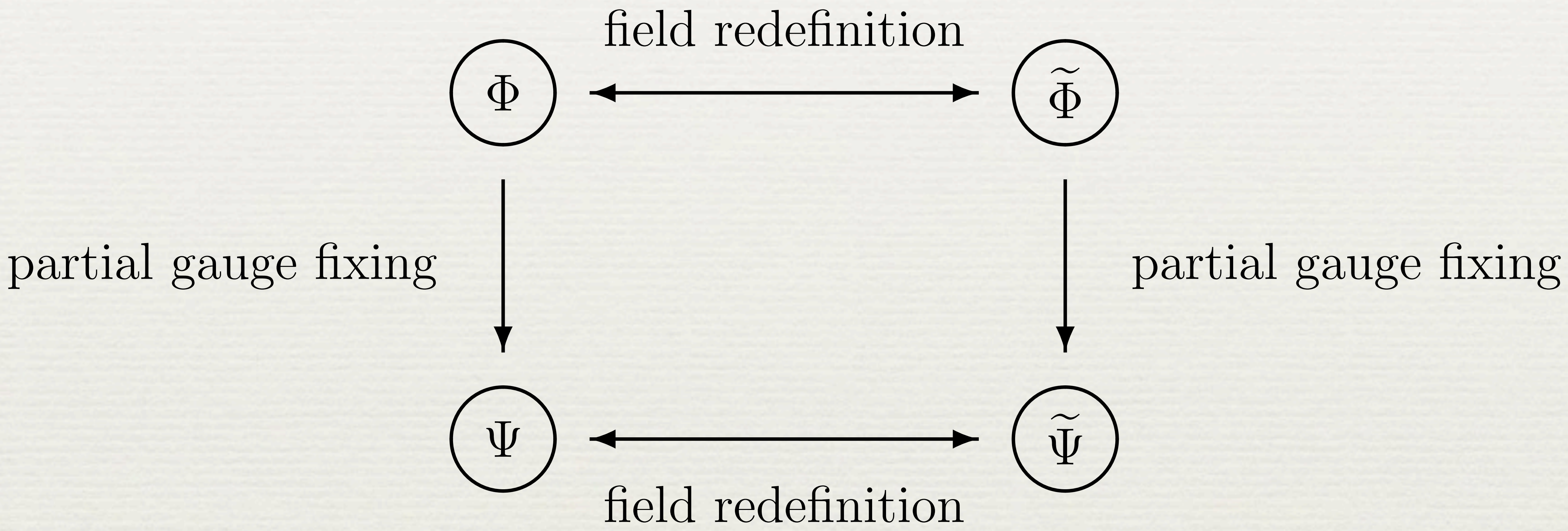
Furthermore, $A_\eta(t)$ and $A_t(t)$ can be interpreted as being obtained from

$$A_\eta(t) = \pi_1 \, \mathbf{G} \, \frac{1}{1 - \eta \, \Phi(t)} \,,$$

$$A_t(t) = \pi_1 \, \mathbf{G} \left(\frac{1}{1 - \eta \, \Phi(t)} \otimes \partial_t \Phi(t) \otimes \frac{1}{1 - \eta \, \Phi(t)} \right)$$

by the partial gauge fixing $\Phi = \xi \Psi$.

The theory with the A_∞ structure can be embedded in a theory based on the large Hilbert space, which is related to the Berkovits formulation by field redefinition.


field redefinition
Φ
$\widetilde{\Phi}$
partial gauge fixing
partial gauge fixing
Ψ
$\widetilde{\Psi}$
field redefinition

Future directions

- Extension to the closed string:
  heterotic string field theory and type II superstring field theory

- The relation to the supermoduli space of super-Riemann surfaces
  Ohmori and Okawa, in preparation.

- Extension to the Ramond sector.